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Painlevé Analysis and Hamiltonian Structure of a New Space-Dependent KdV Equation

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We deduce the Lax pair for a new space-dependent KdV equation, $u_t = -\frac{5}{4}(u_{xxx}+6uu_x)+\eta u_x/x^2+\beta u/x^3$, via the technique of Painlevé analysis. From it, infinitely many conservation laws are deduced and the symplectic structure is obtained.

The extension of the integrable class of nonlinear equations is one of the most important aspects of present research. Of late, various extensions and modification of the usual KdV equation have been suggested (Roy Chowdhury and Mahato, 1982; Nakamura and Chen, 1981). Here we propose a new space-dependent KdV equation and prove its complete integrability via a Painlevé analysis along the lines of Weiss (1983). In the course of our analysis we deduce the Lax pair from this singular point analysis. The deduced Lax pair then yields infinitely many conservation laws, which can be used to deduce the Hamiltonian and the symplectic operator pertaining to the Hamiltonian structure.

We study the equation

$$u_t = -\tfrac{5}{4}(u_{xxx}+6uu_x)+\eta\frac{u_x}{x^2}+\frac{\beta u}{x^3} \tag{1}$$

We set, following Weiss,

$$u = \sum_{j=0}^{\alpha} u_j(x,t)\phi^{\alpha+j} \tag{2}$$

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with $\alpha<0$ and $\phi(x, t)=0$ being the solution manifold. Comparing the leading singular terms, we get $\alpha=-2$, and from the general coefficient of ϕ^{-2+j} we get

$$\begin{aligned} &u_{j+2,t}+(j+1)\phi_t u_{j+3} \\ &= -\tfrac{5}{4}[u_{j+2,xxx}+3(j+1)u_{j+3,xx}\phi_x+3(j+2)(j+1)u_{j+4,x}\phi_x^2 \\ &\quad +3(j+1)u_{j+3,x}\phi_{xx}+(j+3)(j+2)(j+1)u_{j+5}\phi_x^3 \\ &\quad +3(j+2)(j+1)\ u_{j+4}\phi_x\phi_{xx}+(j+1)u_{j+3}\phi_{xxx}] \\ &\quad -\frac{15}{2}\left[\sum_k u_{j+4-k}u_{kx}+\sum_k u_{j+5-k}u_k(k-2)\phi_x\right] \\ &\quad +\frac{\eta}{x^2}[u_{j+2,x}+(j+1)u_{j+3}\phi_x]+\frac{\beta}{x^3}u_{j+2} \end{aligned} \tag{3}$$

Equation (3) leads to

$$u_0=-2\phi_x^2 \tag{4}$$

and the equation for the resonances

$$(r+1)(r-4)(r-6)=0 \tag{5}$$

which is, for $r=-1, 4, 6$, the same as the usual KdV problem. We also have

$$u_1=2\phi_{xx} \tag{6}$$

The arbitrariness of the expansion coefficients at $r=4$ and 6 are easy to verify; we here proceed directly to the major problem of obtaining the Lax pair for the present equation. For this purpose we truncate the WTC series [equation (2)] at the constant level, that is, we set $u_3=u_4=u_5=u_6$, and so on. From this we obtain the consistency condition in the form

$$\begin{aligned} -2u_0\phi_t &= -\tfrac{5}{4}(-6u_{0xx}\phi_x+6u_{1x}\phi_x^2-6u_{0x}\phi_{xx}+6u_1\phi_x\phi_{xx}-2u_0\phi_{xxx}) \\ &\quad -\tfrac{15}{2}(u_1u_{0x}+u_0u_{1x}-2u_2u_0\phi_x-u_1^2\phi_x)-\frac{2\eta}{x^2}u_0\phi_x \end{aligned} \tag{7}$$

and

$$\begin{aligned} u_1\phi_t &= -\tfrac{5}{4}(u_{0xxx}-3u_{1xx}\phi_x-3u_{1x}\phi_{xx}-u_1\phi_{xxx}) \\ &\quad +\frac{\eta}{x^2}(u_{0x}-u_1\phi_x)+\frac{\beta u_0}{x^3}-\tfrac{15}{2}(u_2u_{0x}+u_1u_{1x}+u_0u_{2x}-u_2u_1\phi_2) \end{aligned} \tag{8}$$

Substituting the values of u_1 and u_0, we obtain

$$4\phi_x\phi_t+20\phi_x\phi_{xxx}-15\phi_x^2+30\phi_x^2u_2-\frac{4\eta}{x^2}\phi_x^2=0 \tag{9}$$

$$4\phi_x\phi_{xt}+5\phi_x\phi_{xxxx}-\frac{4\eta}{x^2}\phi_x\phi_{xx}-\frac{4\beta}{x^3}\phi_x^2+30\phi_x\phi_{xx}u_2-\frac{8\eta}{x^2}\phi_x^2=0 \tag{10}$$

and for $r=6$

$$u_{2t}=-\tfrac{5}{4}(u_{2xxx}+6u_2u_{2x})+\frac{\eta}{x^2}u_{2x}+\frac{\beta}{x^3}u^2 \tag{11}$$

So u_2 is again a solution of the same equation and the WTC Expansion can serve as a Backlund transformation. It reads

$$\begin{aligned} u&=u_0\phi^{-2}+u_1\phi^{-1}+u_2\\ &=2\frac{\partial^2}{\partial x^2}\ln\phi+u_2 \end{aligned} \tag{12}$$

Eliminating u_2 from (9) and (10), we get

$$\frac{\phi_t}{\phi_x}+\tfrac{5}{4}\{\phi, x\}=\lambda \tag{13}$$

if η and β are related by $\beta=-2\eta$ and $\eta=15$, where λ is an integration constant. So from (9) we deduce

$$u_2+\frac{2\lambda}{15}=-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x}+\frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}+\frac{2}{x^2} \tag{14}$$

In equation (13), $\{\phi, x\}$ denotes the Schwarztian derivative defined by (Weiss, 1984)

$$\{\phi, x\}=\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right)-\frac{1}{2}\frac{\phi_{xx}^2}{\phi_x^2}$$

If we now set $\phi=v^2$, then (14) at once is converted to

$$v_{xx}=-\left(u_2+\frac{2\lambda}{15}-\frac{2}{x^2}\right)v \tag{15}$$

which is something like a Schrödinger equation with an external field. On the other hand, writing (13) in the form

$$\phi_t=\left[\lambda+\frac{5}{8}\left(u_2+\frac{2\lambda}{15}-\frac{v_x^2}{v^2}-\frac{2}{x^2}\right)\right]\phi_x$$

and differentiating with respect to x, using (15) and $\phi=v^2$, we get

$$v_t=\frac{5}{4}\left(u_2+\frac{14}{15}\lambda-\frac{1}{x^2}\right)v_x+\frac{5}{4}\left(\frac{u_{2x}}{4}+\frac{1}{x^3}\right)v \tag{16}$$

Equations (15) and (16) comprise the Lax pair for equation (1). So in this part we have proved the complete integrability both in the sense of Painlevé and in the sense of being associated with a Lax pair.

Let us now set out to study the asymptotic behavior of (15) as $x \to \alpha$. By assumption, $u_2 \to 0$ as $x \to \alpha$, so for v we set

$$v = \exp[i(2\lambda'/15)^{1/2}x] + \int_{-\alpha}^{x} \chi \, dx' \tag{17}$$

whence the equation for χ reads

$$\chi_x + \chi^2 + 2ik\chi + u_2 - \frac{2}{x^2} = 0 \tag{18}$$

where $k = (2\lambda'/15)^{1/2}$.

Expanding χ in negative powers of k

$$\chi = \sum_{n=0}^{\alpha} (2ik)^{-n} \chi_n \tag{19}$$

and substituting into (18), we get

$$\begin{aligned} \chi_0 &= -\left(u - \frac{2}{x^2}\right) \\ \chi_1 &= u_x - u^2 + \frac{4u}{x^2} + \frac{4}{x^3} - \frac{4}{x^4} \end{aligned} \tag{20}$$

along with

$$\chi_{nx} + \sum \chi_{n-m}\chi_m + \chi_{n+1} + \left(u - \frac{2}{x^2}\right)\delta_{n,0} = 0 \tag{21}$$

which yields an infinite set of conserved quantities. Each of these conserved quantities may serve as Hamiltonian for the system. It is not difficult to observe that equation (1) can be written as

$$u_t = O_1 \frac{\delta \chi_1}{\delta u}$$

where O_1 is a symplectic operator,

$$O_1 = -\frac{5}{8} D^3 + \frac{5}{2}\left(\frac{1}{x} D^2 - D^2 \frac{1}{x}\right) - \frac{5}{8}(2uD + 2Du) \tag{22}$$

Two important differences from the case of the usual KdV equation are to be noted. First, the Hamiltonians are explicitly space dependent and so is the symplectic operator O_1. Equation (22) is similar in form with the second

Hamiltonian form of the usual KdV equation. Finally, before we conclude, we make some observations regarding the symmetry structure of our equation. Equation (1) is scaling invariant with the generator

$$\eta_1 = 2u + xu_x + 3tu_t$$

and it is also invariant under the transformation (Bluman and Cole, 1974) $x^* = x + \varepsilon A$, $t^* = t + \varepsilon B$, $u^* = u - \varepsilon \cdot 4A/x^3$, where A and B are constants. This suggests the transformation

$$u = \frac{2}{x^2} + f(Bx - At) \tag{23}$$

and setting (23) in (1), we get

$$\tfrac{5}{4}(B^3 f_{\xi\xi} + 3Bf^2) = Af + D \tag{24}$$

which is nothing but an ordinary Painlevé class equation (Ince, 1935).

In the above analysis we have suggested a new space-dependent, completely integrable KdV equation, for which all the properties of integrable nonlinear equations are seen to hold.

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